ON THE CONVERGENCE OF CONTINUED FRACTIONS WITH COMPLEX ELEMENTS*

BY

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Up to the present time few theorems of a general character for the convergence of continued fractions with complex elements have been obtained, and these few are of very recent date. In the first section of this paper such theorems upon the subject as are known to the writer are brought together for the purpose of indicating the present state of our knowledge, and the scope of the paper is also explained. Some new criteria for convergence are then deduced in the succeeding sections. The results obtained are summed up in theorems 1–10, which may be read independently of the rest of the paper. The demonstration of these theorems is based upon certain equations, Nos. 3–8, 11, and 12, which seem to be new and of a fundamental character.

I. *Summary of previous criteria for convergence.*

§ 1. Two classes of continued fractions

$$\frac{\mu_1}{\lambda_1 +} \frac{\mu_2}{\lambda_2 +} \frac{\mu_3}{\lambda_3 +} \cdots$$

with real elements are commonly treated in mathematical writings. In the first class all the elements are positive; in the second the numerators of the partial quotients are negative and the denominators positive.

Complete criteria have been obtained for the convergence and divergence of continued fractions of the first kind. If, namely, the fraction is thrown into the form

$$\frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \qquad (a_n \geqq 0),$$

it will converge when $\sum a_n$ is divergent; on the other hand, if $\sum a_n$ is convergent, the even convergents have one limit and the odd convergents have another. No criteria of equal generality have been obtained for the second class of con-

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tinued fractions. The principal theorem, however, establishes the convergence of the fraction when $\lambda_n \geqq 1 + |\mu_n|$.*

§ 2. The convergence-theorems for continued fractions with imaginary elements may be regarded as generalizations of these theorems for real continued fractions. In 1898 PRINGSHEIM † showed that the condition $|\lambda_n| \geqq 1 + |\mu_n|$ was sufficient to ensure the convergence not only of continued fractions with real elements but also of those containing complex elements. If, furthermore, the continued fraction has the customary normal form in which $\mu_n = 1$, this condition may be replaced by the less restricted one:

$$\frac{1}{|\lambda_{2n-1}|} + \frac{1}{|\lambda_{2n}|} \leqq 1.$$

In this modified form PRINGSHEIM'S criterion is applicable to the continued fractions of HURWITZ,‡ in which the partial denominators are complex integers.

The only extension of the convergence-theorem for the first class of real continued fractions is contained in a remarkable and extended memoir by STIELTJES, § published in 1894. STIELTJES considers the fraction

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \cdots \qquad (a_n > 0, \quad n = 1, 2, 3, \cdots),$$

and shows that when $\sum a_n$ is divergent, it converges over the entire plane of z with the exception of the negative half of the real axis. On the other hand, when $\sum a_n$ is convergent, the even and the odd convergents separately converge, the limits of their numerators and denominators being holomorphic functions of *genre* 0 whose roots all lie upon the negative half of the axis.

From a function-theoretical standpoint this result is of great importance. It also calls attention anew to the fact that algebraic continued fractions commonly have a more extended region of convergence than the corresponding infinite series, and hence have, at least prospectively, a wide field of usefulness. The convergence-proofs of STIELTJES are most admirable, both by themselves and in relation to the theory of functions. When, however, they are considered solely with respect to the general theory of continued fractions, they are not entirely satisfactory. Not only are his proofs based upon the properties of the conver-

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* For reference to the possibilities when this condition is not fulfilled, see PRINGSHEIM'S report in the *Encyklopädie der mathematischen Wissenschaften.*

† Sitzungsberichte der mathematisch-physikalischen Classe der Münchener Akademie, vol. 28, p. 295.

‡ Acta Mathematica, vol. 11.

§ Annales de la Faculté des Sciences de Toulouse, vol. 8, with continuation in vol. 9.

gents as functions of z,* but the form of his continued fraction is also very restricted, inasmuch as the ratio of the real to the imaginary component is the same in all the denominators of the alternate partial quotients.

These considerations have led to the present paper. It comprises such results as have suggested themselves in seeking to extend the convergence-theorems for the first class of real continued fractions to complex continued fractions.

To complete the historical review, mention should be made of the theorems of PRINGSHEIM and STIELTJES relating to continued fractions in which the denominators of the partial quotients are equal to unity. PRINGSHEIM† proves that such a fraction will converge if

$$|\mu_2| < \tfrac{1}{2}, \quad |\mu_{2n+2}| + |\mu_{2n+1}| \leqq \tfrac{1}{2} \qquad (n = 1, 2, \cdots).$$

STIELTJES‡ treats the fraction

$$\frac{b_1 z}{1} + \frac{b_2 z}{1} + \frac{b_3 z}{1} + \cdots \qquad (b_n > 0),$$

and shows that it represents a meromorphic function when b_n approaches zero as its limit, and only then.

Various other theorems pertaining to continued fractions are reducible to the foregoing by transformation of the fraction.

II. *Fundamental equations.*

§ 3. Let

$$\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} + \cdots$$

be any continued fraction whose elements are complex numbers. Denote by p_n and q_n the numerator and denominator of the n-th convergent, by M_n the modulus of q_n. If then we put

$$\lambda_n = a_n + i\beta_n, \quad \mu_n = \gamma_n + i\delta_n,$$

$$p_n = r'_n + is'_n, \quad q_n = r_n + is_n,$$

the familiar laws for the formation of the successive convergents,

$$(1) \qquad p_n = \lambda_n p_{n-1} + \mu_n p_{n-2}, \quad q_n = \lambda_n q_{n-1} + \mu_n q_{n-2},$$

* See, for example, the remarkable preliminary theorem which he establishes in order to prove the convergence of the continued fraction when $\sum a_n$ is divergent and the real part of z is negative. Loc. cit., vol. 8, p. J56.

† Sitzungsberichte der Müncheuer Akademie, vol. 28.

‡ Annales de Toulouse, vol. 9, p. A, 42–47.

may be replaced by

$$
(2) \qquad \begin{aligned} r_n &= a_n r_{n-1} + \gamma_n r_{n-2} - \beta_n s_{n-1} - \delta_n s_{n-2}, \\ s_n &= a_n s_{n-1} + \gamma_n s_{n-2} + \beta_n r_{n-1} + \delta_n r_{n-2}, \end{aligned}
$$

with two similar equations for r_n' and s_n'.

Squaring and adding equations (2) we obtain next the following fundamental relation between the moduli of three consecutive denominators:

$$
(3) \qquad \begin{aligned} M_n^2 = (a_n^2 + \beta_n^2) M_{n-1}^2 + (\gamma_n^2 + \delta_n^2) M_{n-2}^2 \\ + 2(a_n\gamma_n + \beta_n\delta_n) C_{n-1} + 2(\beta_n\gamma_n - a_n\delta_n) D_{n-1}, \end{aligned}
$$

in which

$$
(4) \qquad C_{n-1} = r_{n-1} r_{n-2} + s_{n-1} s_{n-2},
$$

$$
(5) \qquad D_{n-1} = r_{n-1} s_{n-2} - s_{n-1} r_{n-2}.
$$

The form of the last two equations may be modified advantageously by introducing in place of r_{n-1} and s_{n-1} the expressions which are obtained by the substitution of $n-1$ for n in (2). The equations then become

$$
(6) \qquad C_{n-1} = a_{n-1} M_{n-2}^2 + \gamma_{n-1} C_{n-2} - \delta_{n-1} D_{n-2},
$$

$$
(7) \qquad D_{n-1} = -\beta_{n-1} M_{n-2}^2 - \gamma_{n-1} D_{n-2} - \delta_{n-1} C_{n-2}.
$$

In this form they may be used with (3) to compute M_{n-1}, C_{n-1}, D_{n-1} from the corresponding quantities of next lower subscript. If also we take

$$
r_0 = 1, \quad r_0' = s_0 = s_0' = 0,
$$

we shall have

$$
M_0^2 = 1, \quad C_1 = r_1 = a_1 M_0^2, \quad D_1 = -s_1 = -\beta_1 M_0^2.
$$

Hence C_{n-1}, D_{n-1}, M_{n-1}^2 are each homogeneous linear functions of the squares of the moduli M_i with lower subscripts (ultimately of M_0^2 and M_1^2), in which the coefficients are entire functions of a_i, β_i, γ_i, $\delta_i (i = 1, 2, \cdots, n-1)$.

From (4) and (5) we also get the important relation:

$$
(8) \qquad C_{n-1}^2 + D_{n-1}^2 = M_{n-1}^2 M_{n-2}^2.
$$

§ 4. To investigate the convergence of a continued fraction recourse is sometimes had to the equation:*

$$
(9) \qquad \begin{aligned} \frac{p_{n+m}}{q_{n+m}} = \frac{p_n}{q_n} + (-1)^n \mu_1 \mu_2 \cdots \mu_{n+1} \biggl(\frac{1}{q_n q_{n+1}} - \frac{\mu_{n+2}}{q_{n+1} q_{n+2}} \\ + \frac{\mu_{n+2}\mu_{n+3}}{q_{n+2} q_{n+3}} + \cdots + (-1)^{m-1} \frac{\mu_{n+2}\mu_{n+3}\cdots\mu_{n+m}}{q_{n+m-1} q_{n+m}} \biggr), \end{aligned}
$$

* HEINE'S *Kugelfunctionen*, vol. 1, p. 263.

in which m is allowed to increase indefinitely. The convergence of a continued fraction is thus made to depend upon that of an infinite series. As, however, this form of the series is not well adapted to the consideration of cases of alternating convergence, it will be better to group by twos the terms in the parenthesis of (9). With the aid of the relation

$$\frac{1}{q_n q_{n+1}} - \frac{\mu_{n+2}}{q_{n+1} q_{n+2}} = \frac{\lambda_{n+2}}{q_n q_{n+2}},$$

the series may then be thrown into the form:

$$(10) \quad \frac{p_n}{q_n} + (-1)^n \mu_1 \mu_2 \cdots \mu_{n+1} \left(\frac{\lambda_{n+2}}{q_n q_{n+2}} + \frac{\lambda_{n+4} \mu_{n+2} \mu_{n+3}}{q_{n+2} q_{n+4}} + \frac{\lambda_{n+6} \mu_{n+2} \mu_{n+3} \cdots \mu_{n+5}}{q_{n+4} q_{n+6}} + \cdots \right).$$

Two different cases arise according as n is even or odd. If the series converges in both cases and the continued product of the μ_i is convergent and greater than zero, *the necessary and sufficient condition for the coincidence of the two limiting values of the series is that $M_{n-1} M_{n-2}$ shall increase indefinitely with n.* The continued fraction then converges in the ordinary sense of the term. On the other hand, if $M_{n-1} M_{n-2}$ has a finite limit, the convergence is of an alternating character.

The series (10) has the advantage over (9) that it is frequently absolutely convergent when the latter is only conditionally convergent.

Either series loses its significance if q_i vanishes in one or more of its terms. Should this, however, happen only a finite number of times, the continued fraction may still converge, for the objectionable terms can be removed from the series by increasing n sufficiently.

In one other particular the convergence of a continued fraction requires special comment. The removal of the first r partial quotients μ_i/λ_i may change a convergent continued fraction into a divergent one, or *vica versa.* Adopting a term proposed by PRINGSHEIM,* we shall say that the convergence is *unconditional* when it is not affected by the removal of any finite number of partial quotients, beginning with the first.

§ 5. Besides discussing the convergence of the fraction, we shall in certain cases determine the signs of its real and imaginary parts. For this purpose let p_n/q_n be thrown into the form:

$$\frac{r_n r'_n + s_n s'_n + i(r_n s'_n - r'_n s_n)}{M_n^2}.$$

* For a further discussion of unconditional convergence consult the memoir of PRINGSHEIM previously cited.

If we then place

$$K_n = r_n r'_n + s_n s'_n,$$

$$L_n = r_n s'_n - r'_n s_n,$$

and combine these equations with (2) and the two analogous equations for r'_n and s'_n, we obtain

$$\text{(11)}\quad \begin{aligned} K_n = (a_n^2 + \beta_n^2) K_{n-1} + (\gamma_n^2 + \delta_n^2) K_{n-2} + 2(a_n\gamma_n + \beta_n\delta_n) k_{a\gamma}^{(n-1)} \\ + 2(\beta_n\gamma_n - a_n\delta_n) k_{\beta\gamma}^{(n-1)}, \end{aligned}$$

$$\text{(12)}\quad \begin{aligned} L_n = (a_n^2 + \beta_n^2) L_{n-1} + (\gamma_n^2 + \delta_n^2) L_{n-2} + 2(a_n\gamma_n + \beta_n\delta_n) l_{a\gamma}^{(n-1)} \\ + 2(\beta_n\gamma_n - a_n\delta_n) l_{\beta\gamma}^{(n-1)}, \end{aligned}$$

in which

$$k_{a\gamma}^{(n-1)} = \tfrac{1}{2}(r_{n-1}r'_{n-2} + r'_{n-1}r_{n-2} + s_{n-1}s'_{n-2} + s'_{n-1}s_{n-2}),$$

$$k_{\beta\gamma}^{(n-1)} = \tfrac{1}{2}(r_{n-1}s'_{n-2} + r'_{n-1}s_{n-2} - s_{n-1}r'_{n-2} - s'_{n-1}r_{n-2}),$$

$$l_{a\gamma}^{(n-1)} = \tfrac{1}{2}(r_{n-1}s'_{n-2} - r'_{n-1}s_{n-2} - s_{n-1}r'_{n-2} + s'_{n-1}r_{n-2}),$$

$$l_{\beta\gamma}^{(n-1)} = \tfrac{1}{2}(-r_{n-1}r'_{n-2} + r'_{n-1}r_{n-2} - s_{n-1}s'_{n-2} + s'_{n-1}s_{n-2}).$$

The last set of equations may be simplified in the same manner as (4) and (5). They then become

$$k_{a\gamma}^{(n-1)} = a_{n-1}K_{n-2} + \gamma_{n-1}k_{a\gamma}^{(n-2)} - \delta_{n-1}k_{\beta\gamma}^{(n-2)},$$

$$k_{\beta\gamma}^{(n-1)} = -\beta_{n-1}K_{n-2} - \gamma_{n-1}k_{\beta\gamma}^{(n-2)} - \delta_{n-1}k_{a\gamma}^{(n-2)},$$

$$l_{a\gamma}^{(n-1)} = a_{n-1}L_{n-2} + \gamma_{n-1}l_{a\gamma}^{(n-2)} - \delta_{n-1}l_{\beta\gamma}^{(n-2)},$$

$$l_{\beta\gamma}^{(n-1)} = -\beta_{n-1}L_{n-2} - \gamma_{n-1}l_{\beta\gamma}^{(n-2)} - \delta_{n-1}l_{a\gamma}^{(n-2)},$$

and may be used in conjunction with (11) and (12) to find K_n and L_n for successive values of n.

It will, of course, be observed that the above equations are similar in their construction to equations 3–7.

III. *On the convergence and properties of a certain class of continued fractions.*

§ 6. Our attention in this section will be confined entirely to continued fractions which have been reduced to the normal form for which $\gamma_n = 1$, $\delta_n = 0$. Equations (3), (6), and (7) then take the simpler form:

$$\text{(13)}\quad M_n^2 = (a_n^2 + \beta_n^2) M_{n-1}^2 + M_{n-2}^2 + 2a_n C_{n-1} + 2\beta_n D_{n-1},$$

$$\text{(14)}\quad C_{n-1} = a_{n-1}M_{n-2}^2 + C_{n-2},$$

$$\text{(15)}\quad D_{n-1} = -\beta_{n-1}M_{n-2}^2 - D_{n-2},$$

and the series (10) simultaneously becomes

$$(16) \qquad \frac{p_n}{q_n} + (-1)^n \left(\frac{a_{n+2} + i\beta_{n+2}}{q_n q_{n+2}} + \frac{a_{n+4} + i\beta_{n+4}}{q_{n+2} q_{n+4}} + \cdots \right).$$

We shall first determine what signs for a_n and β_n are most favorable to the absolute convergence of (16), supposing their absolute values to be given. Since

$$(17) \qquad C_{n-1} = \sum_{i=1}^{n-1} a_i M_{i-1}^2, \quad D_{n-1} = -\sum_{i=1}^{n-1} (-1)^{n-i+1} \beta_i M_{i-1}^2,$$

C_{n-1} will increase in absolute value with increasing n if the a_n have a common sign, and D_{n-1} will similarly increase if the β_n are alternately of opposite sign (each zero value of β_n being included and reckoned with an appropriate sign). Moreover $a_n C_{n-1}$ and $\beta_n D_{n-1}$ will then never be negative. These conditions are clearly as favorable as possible for the increase of $M_n = |q_n|$ and hence also for the absolute convergence of (16).

§ 7. In considering the convergence of the continued fraction two cases are to be distinguished according as $\sum |\lambda_n|$—that is, $\sum |a_n + i\beta_n|$—if convergent or divergent.

Consider first the former case and suppose either that the a_n have a common sign or that the β_n alternate in sign. It is immaterial which supposition is made because a_n and $-\beta_n$ enter in like manner into the fundamental equations. In either case we have from (1):

$$M_m \leqq |\lambda_m| M_{m-1} + M_{m-2} \leqq (1 + |\lambda_m|) M',$$

in which M' is used to represent the larger of the two moduli M_{m-1}, M_{m-2}. This shows that

$$M_n < M' \prod_{i=m}^{n} (1 + |\lambda_i|) \qquad (n > m).$$

But by a well-known theorem $\prod (1 + |\lambda_i|)$, $(i = 1, 2, \cdots)$, converges with $\sum |\lambda_i|$. There must accordingly be an upper limit to the value of M_n. On the other hand, by virtue of the hypothesis concerning a_n or β_n, either $|C_{n-1}|$ or $|D_{n-1}|$ will increase with n, or at least not decrease. It follows then from (8) that $M_{n-1} M_{n-2}$ has a lower limit. This can only be if M_n, which has an upper limit, has also a lower limit.

Turning next to the series (16) we see now that it must be absolutely convergent, since the numerators of its terms form an absolutely convergent series while the moduli of the denominators have a lower limit. As also $M_{n-1} M_{n-2}$ does not indefinitely increase, two distinct values for the series will be obtained according as n is odd or even. We have therefore the following result:

THEOREM I. *In any continued fraction*

$$\frac{1}{a_1 + i\beta_1} + \frac{1}{a_2 + i\beta_2} + \frac{1}{a_3 + i\beta_3} + \cdots \tag{18}$$

the even and the odd convergents will both converge and their limits will be distinct if (1) $\sum |a_n + i\beta_n|$ *is convergent and* (2) *either the a-constituents have a common sign or the β-constituents alternate in sign.**

§ 8. In § 7 it was shown, and without making use of condition (1), that $M_{n-1}M_{n-2}$ had a lower limit. An immediate inference from this is that M_n cannot vanish for any finite value of n. A like conclusion obviously holds for the numerator of the nth convergent, inasmuch as the equations for the formation of the successive numerators are similar to those for the denominators of the convergents. We have thus a very simple proof of the following theorem:

THEOREM 2. *Neither the numerator nor the denominator of any convergent of the continued fraction* (18) *will vanish if the a-constituents have a common sign or if the β-constituents alternate in sign.*

§ 9. We pass now to the consideration of the case in which $\sum |a_n + i\beta_n|$ diverges. Let it be first supposed that both a_n and β_n are restricted in sign in the manner indicated in the preceding theorem. Every term in (13) will then be positive or zero, and M_n must be equal to or greater than M_{n-2} for every value of n. Accordingly M_n has a lower limit. Furthermore, by our initial hypothesis either $\sum |a_n|$ or $\sum |\beta_n|$ must diverge, or both. It follows therefore from (17) that at least one of the two quantities, C_{n-1} and D_{n-1} increases indefinitely with n. But by (8) $M_{n-1}M_{n-2}$ must increase in the same manner. If, therefore, the series (16) converges, it must have the same limit whether n be odd or even.

To prove that (16) does in fact converge, we shall establish separately the convergence of the two series:

$$\sum_{n=1}^{\infty} \frac{|a_n|}{M_{n-2}M_n}, \quad \sum_{n=1}^{\infty} \frac{|\beta_n|}{M_{n-2}M_n}.$$

Consider first the former. If in computing M_n by formula (13) for successive values of n we should equate each β_n to zero, smaller values for M_n would be obtained, which we will denote by M'_n. Obviously these are successively connected by the relation:

$$M'_n = |a_n| M'_{n-1} + M'_{n-2},$$

or by its equivalent

$$\frac{1}{M'_{n-2}M'_{n-1}} = \frac{|a_n|}{M'_{n-2}M'_n} + \frac{1}{M'_{n-1}M'_n}.$$

* Zero-constituents are admissible in this and the following theorems, being taken as having the signs which belong to the places in which they are found.

$$\frac{1}{M'_{m-2}M'_{m-1}} \geqq \sum_{n=m}^{\infty} \frac{|a_n|}{M'_{n-2}M'_n}.$$

This proves that

$$\sum \frac{|a_n|}{M'_{n-2}M'_n}$$

converges, and hence also $\sum |a_n| M_{n-2}^{-1} M_n^{-1}$. Similar considerations apply to prove the convergence of $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$.

We conclude, therefore, that the continued fraction has a definite limit, and the reasoning evidently holds if the first n partial quotients are omitted. This result may be restated in

THEOREM 3. *If the a-constituents in* (18) *all have the same sign and the β-constituents alternate in sign, the continued fraction will converge unconditionally when $\sum |a_n + i\beta_n|$ is divergent.*

§ 10. The signs of the real and imaginary parts of the convergents can be quickly determined. Equations 13–15 hold when M_i^2, C_i, D_i are replaced by K_i, $k_{a\gamma}^{(i)}$, $k_{\beta\gamma}^{(i)}$ or by L_i, $l_{a\gamma}^{(i)}$, $l_{\beta\gamma}^{(i)}$. The initial values of these quantities are as follows:

$$K_0 = L_0 = 0, \quad K_1 = a_1, \quad L_1 = -\beta_1,$$

$$k_{a\gamma}^{(1)} = l_{\beta\gamma}^{(1)} = \tfrac{1}{2}, \quad k_{\beta\gamma}^{(1)} = l_{a\gamma}^{(1)} = 0.$$

An easy mathematical induction then shows that all the terms in (13) after either of the above substitutions have a common sign. From this we get at once

THEOREM 4. *If the a-constituents have a common sign and the β-constituents alternate in sign, the real part of any convergent will have the same sign as a_1 and the imaginary part the same sign as $-\beta_1$.*

§ 11. Returning now to the continued fraction of theorem 3, we proceed to consider the effect of the admission of a finite number of a-constituents or of β-constituents which fail to satisfy the conditions there imposed. It is immaterial whether the exceptional elements occur among the a_n or the β_n; let them occur among the former.

When n is increased, there must be a value—call it m—subsequent to which a_n and C_{n-1} have each an invariable sign. If they have the same sign, all the steps of the proof of theorem 3 will apply after the m-th convergent. This will necessarily be the case when $\sum |a_n| M_{n-1}^2$ $(n = 1, 2, \cdots)$ is divergent.

There remains for consideration only the special case in which $\sum |a_n| M_{n-1}^2$ converges and a_n and C_{n-1} have opposite signs when n exceeds m. Equation (13) then gives

$$M_n^2 \geqq a_n^2 M_{n-1}^2 - 2\,|a_n C_{n-1}| + M_{n-2}^2,$$

whence by combination with (8) we obtain

$$M_n^2 \geqq M_{n-2}^2 \left(1 - \frac{|a_n C_{n-1}| M_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2}\right)^2. \tag{19}$$

Now since $\sum |a_n| M_{n-1}^2$ converges, $|C_{n-1}|$ has an upper limit. On the other hand, $|D_{n-1}|$, which never diminishes, will have a finite lower limit after some fixed value of n, unless it is equal to zero for every value of n. This exceptional case will not occur if there is an infinite number of values of n for which $|\beta_n| > 0$, as we will suppose, neither can it occur if there is a single such value greater than m, inasmuch as $M_{m+i} > 0$. An upper limit must therefore exist for $|C_{n-1}|/(C_{n-1}^2 + D_{n-1}^2)$. Since also $|a_n| M_{n-1}^2$ diminishes indefinitely with increasing n, the second term of the binomial in the last inequality may be made as small as desired. Let m be taken large enough to ensure that it shall always be less than unity when $n > m$. We then have

$$M_{m+2n} > M_m \prod_i \left(1 - \frac{|a_i C_{i-1}| M_{i-1}^2}{C_{i-1}^2 + D_{i-1}^2}\right) \quad (i = m+2,\, m+4,\, \cdots,\, m+2n).$$

If in the last expression n is indefinitely increased, the product $\prod$ must be convergent since $\sum |a_i M_{i-1}^2|$ is convergent and $|C_{i-1}|/(C_{i-1}^2 + D_{i-1}^2)$ has an upper limit. Moreover, by a well-known theorem the limiting value of $\prod$ must be greater than zero. Hence M_{m+2n} has a lower limit, and the same is true of M_n since m may be taken to be either odd or even.

The convergence of $\sum |a_n|$ may next be quickly argued. This follows from the fact that the terms in the convergent series $\sum a_n M_{n-1}^2$ have ultimately a common sign while M_n has a lower limit. Now by our original hypothesis $\sum |a_n + i\beta_n|$ was a divergent series. This necessitates that $\sum |\beta_n|$ should likewise diverge. But when this diverges, $|D_{n-1}|$ increases indefinitely and with it simultaneously $M_{n-1} M_{n-2}$.

To demonstrate the convergence of the continued fraction, it remains only to prove that (16) converges. This can be quickly done. For, on the one hand, $\sum |a_n| M_{n-2}^{-1} M_n^{-1}$ obviously converges. On the other hand, if each a_n is replaced by zero, $\sum |\beta_n| M_{n-2}^{-1} M_n$ will converge, as we saw in § 9. But from (19) it is clear that the presence of the a-terms in (13), when it does not actually increase the value of M_n, will cut it down in a ratio not exceeding

$$1 : \left(1 - \frac{|a_n C_{n-1}| M_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2}\right).$$

It has also been shown that the product of all such ratios for which n exceeds m is convergent. The restoration of the a-terms will therefore not reduce the If in the latter we set in turn $n = m,\, m+1,\, \cdots$, and then add, we get value of $M_n (n > m)$ by more than a certain fixed part of itself. Since this in no wise affects the convergence of $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$, the series must still converge. This completes the proof.

The conclusion reached can be summed up as follows:

THEOREM 5. *The continued fraction of theorem* 3 *will still converge if a finite number of the* a_n *or of the* β_n *are admitted which fail to satisfy the conditions there imposed, provided that after the last irregularity there is at least one value of* n *for which* β_n, *respectively* a_n *is not equal to zero.*

§ 12. The criterion of theorem 5 differs from most or all criteria hitherto given for the convergence of continued fractions in that it admits a finite number of irregularities whose positions in the continued fraction are entirely arbitrary. This raises the question whether an infinite number of exceptional values of a_n might not have been admitted into the continued fraction, or, in other words, whether the condition imposed upon β_n was not in itself sufficient to ensure the convergence of the continued fraction, as in the case when $\sum |a_n + i\beta_n|$ converged. That this, however, is in no wise true, can be shown by constructing divergent fractions in which all the β-constitutents fulfill the conditions imposed in the theorem.

Consider, for example, a continued fraction in which β_n is equal to zero for an infinite number of values of n but not for all values. When $\beta_n = 0$, we have

$$M_n^2 = \left(M_{n-2} + \frac{a_n C_{n-1}}{M_{n-2}} \right)^2 + \frac{a_n^2 D_{n-1}^2}{M_{n-2}^2}.$$

Let now a_n be so chosen that the term in the parenthesis shall vanish. This makes $|D_{n-1}^{-1}| = |a_n| M_{n-2}^{-1} M_n^{-1}$. If the values of the β_n which do not vanish fulfill the requirements concerning their signs and are also so chosen that $|D_{n-1}|$ shall have an upper limit, the terms of (16) cannot all decrease indefinitely with increasing n. The series (16) and the continued fraction are in consequence divergent.

§ 13. The restriction upon the signs of a_n in theorem 3 may, however, be dispensed with if we assume that $|\beta_n| / |a_n|$ has a finite lower limit. The proof of the convergence of the continued fraction is then not so simple.

The first step will be to show that $|D_n|$ increases indefinitely with n. Suppose, if possible, that the contrary is the case. It will be recalled that the signs of the β_n were so restricted that $|D_n| = \sum |\beta_n| M_{n-1}^2$. If $\sum |\beta_n| M_{n-1}^2$ converges, $\sum |a_n| M_{n-1}^2$ must likewise converge in consequence of the above assumption. Now it was shown in § 11 that when the latter converged, M_n had a finite lower limit. It follows that $\sum |\beta_n|$ should be convergent. But, on the other hand, the assumption of a lower limit for $|\beta_n|/|\alpha_n|$ and the hypothesis that $\sum|\alpha_n + i\beta_n|$ diverges, together necessitate the divergence of $\sum|\beta_n|$. This involves a contradiction. We conclude, therefore, that $|D_n|$ must increase indefinitely, as stated. Equation (8) shows that $M_{n-1}M_{n-2}$ also increases without limit.

To establish the convergence of (16) it clearly suffices to demonstrate the convergence of $\sum|\beta_n|M_{n-2}^{-1}M_n^{-1}$. Let k^{-1} be a lower limit to the value of $|\beta_n|/|\alpha_n|$. Equations (17) show at once that $|C_{n-1}| \leqq k|D_{n-1}|$, whence it follows by (8) that

$$|D_{n-1}| > \frac{M_{n-2}M_{n-1}}{k+1}.$$

When α_n and C_{n-1} are of like sign for any value of n, we have from (13),

$$M_n^2 > \beta_n^2 M_{n-1}^2 + 2\beta_n D_{n-1} + M_{n-2}^2.$$

The combination of this inequality with the preceding gives

$$M_n > \frac{|\beta_n|\,M_{n-1}}{k+1} + M_{n-2}, \tag{20}$$

or

$$\frac{1}{M_{n-2}M_{n-1}} > \frac{|\beta_n|}{(k+1)M_{n-2}M_n} + \frac{1}{M_{n-1}M_n}. \tag{21}$$

The last inequality will also hold when α_n and C_{n-1} have opposite signs if the expression $\alpha_n^2 M_{n-1}^2 + 2\alpha_n C_{n-1}$ in (13) is positive. The only possible exceptions therefore occur when α_n differs in sign from C_{n-1} and

$$|\alpha_n| < \frac{2\,|C_{n-1}|}{M_{n-1}^2}.$$

In this case, if we put

$$\alpha_n = -(1 + x_n)\frac{C_{n-1}}{M_{n-1}^2}, \tag{22}$$

the absolute value of x_n will be less than unity. The combination of the last equation with

$$C_n = -\alpha_n M_{n-1}^2 + C_{n-1}$$

then gives

$$C_n = -x_n C_{n-1}.$$

Hence the necessary and sufficient condition that $\alpha_n^2 M_{n-1}^2 + 2\alpha_n C_{n-1}$ *shall be negative is that* $|C_n|$ *shall be less than* $|C_{n-1}|$.

Even when this condition is fulfilled, the inequality (21) may yet hold. This will, for example, be the case whenever $|C_{n-1}|$ falls below a certain fixed part of $|D_{n-1}|$. For suppose $C_{n-1} < k'|D_{n-1}|$, where k' denotes the positive number less than unity which satisfies the equation

$$\frac{1-kk'}{1+k'}=\frac{1}{1+k}.$$

Then simultaneously

$$|a_n|<k|\beta_n|, \qquad |D_{n-1}|>\frac{M_{n-2}M_{n-1}}{1+k'}.$$

Since also $kk'<1$, it follows that

$$2(a_nC_{n-1}+\beta_nD_{n-1})>2|\beta_nD_{n-1}|(1-kk')>\frac{2|\beta_n|M_{n-1}M_{n-2}}{1+k}.$$

The combination of the last inequality with (13) shows immediately that (20) and (21) also hold.

When for any value of n the inequality (21) is not valid, we shall substitute the expression (22) for a_n in equation (13). This equation then takes the form:

$$M_n^2=\beta_n^2M_{n-1}^2+2\beta_nD_{n-1}+M_{n-2}^2-\frac{(1-x_n^2)\,C_{n-1}^2}{M_{n-1}^2}.$$

Since also

$$\frac{C_{n-1}^2}{M_{n-1}^2}=\frac{C_{n-1}^2}{C_{n-1}^2+D_{n-1}^2}M_{n-2}^2,$$

this gives

$$M_n^2>\beta_n^2M_{n-1}^2+\frac{2|\beta_n|}{k+1}M_{n-1}M_{n-2}+y_n^2M_{n-2}^2,$$

in which

$$y_n=\sqrt{1-\frac{(1-x_n^2)\,C_{n-1}^2}{C_{n-1}^2+D_{n-1}^2}}. \tag{23}$$

It follows then *a fortiori* that

$$\frac{1}{M_{n-2}M_{n-1}}>\frac{|\beta_n|}{(k+1)M_{n-2}M_n}+\frac{y_n}{M_{n-1}M_n}. \tag{24}$$

§ 14. For every value of n there is, then, an inequality of the form (21) or (24). Let these inequalities be arranged in order for successive values of n. When thus arranged they may be divided into sets according to the value of $|D_n|$ in the following manner: Let c be any fixed value greater than unity. Denote also by n_2 the first value of n for which $|D_n|\geqq c|D_1|$, by n_3 the first succeeding value for which $|D_n|\geqq c^2D_1$, etc. Then the first set is to comprise all the inequalities which precede the one in which $n=n_2$; the second set is to begin with the latter inequality and to continue until $n=n_3$, and so on.

Since $|D_n|<M_{n-1}M_n$, the left hand member of the first inequality of the $(r+1)$-th set can not be greater than $1/|D_1|c^r$, and the sum of all such left hand members will therefore be less than a convergent geometrical progression. Consider now any set by itself, and let each inequality contained therein be

multiplied by the product of the y_n which occur in the preceding inequalities of the set. If the inequalities thus modified are added together, each left hand member except the first will be canceled by the last term on the right hand side of the preceding inequality. The sum of the uncanceled left hand members will accordingly be less than the above geometrical progression.

It will be shown immediately that the product of all the y_n which occur in any set has a finite lower limit L independent of the number r which specifies the set. Granting this to be true, if we add together the various sets of inequalities, the sum of all the first terms on the right hand side of the inequalities will be greater than

$$\frac{L}{k+1}\sum_{n=1}^{\infty}\frac{|\beta_n|}{M_{n-2}M_n}.$$

As this sum is less than the above geometric progression, $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$ has a finite limit. The same must be true of the series (16). Since also $M_{n-2}M_{n-1}$ increases indefinitely, the limit will be the same whether n is odd or even. The continued fraction will therefore converge.

It remains now only to establish the existence of the lower limit L. Let $m_1, m_2, m_3, \cdots$ denote in order the values of n which specify for any set the inequalities which are of the form (24). From inspection of (23) it is clear that the product of the y_n will have a finite lower limit if

$$(25) \qquad \sum \frac{(1-x_n)^2 C_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2} \qquad (n = m_1, m_2, m_3, \cdots)$$

has an upper limit. Since $|D_n|$ never diminishes,

$$C_{n-1}^2 + D_{n-1}^2 > D_{m_1-1}^2$$

for each term of this sum, while in the first term

$$|C_{m_1-1}| < k' \, |D_{m_1-1}|.$$

We have also previously seen that $C_n = -x_n C_{n-1}$ for each of the above values of n. Hence if the ratio

$$|C_{m_{r+1}-1}| : |C_{m_r}|$$

is denoted by κ_r, the series (25) will term by term be less than

$$(k')^2\{(1-x_{m_1}^2) + (1-x_{m_2}^2)x_{m_1}^2\kappa_1^2 + (1-x_{m_3}^2)x_{m_1}^2x_{m_2}^2\kappa_1^2\kappa_2^2 + \cdots\}.$$

The expression here in parenthesis obviously has a value not greater than 1 when each κ_n is placed equal to unity. Hence its original value can not exceed the product of all the κ_n^2 which are greater than unity. Our problem is thus reduced to showing that this product has a common upper limit for the various sets of inequalities. This can be quickly proved. For when $|C_n|$ increases from $|C_{m_r}|$ to $|C_{m_{r+1}-1}| \equiv \kappa_r\,|C_{m_r}|$, an increment is given to $|D_n|$ which is at least equal to the increment of $|C_n|$ divided by k. Since also for each of the values of n under consideration $|C_n|$ can not fall below $k'\,|D_n|$, the sum of the increments given to $|D_n|$ must exceed

$$\frac{k'}{k}\,|D_{m_1}|\sum(\kappa_r - 1).$$

On the other hand, as soon as this sum reaches $c\,|D_{m_1}|$, or before, a new set of inequalities must be begun. Hence for each set

$$\sum(\kappa_r - 1) < \frac{kc}{k'}.$$

But the product $\Pi\kappa_r$ has an upper limit simultaneously with $\sum(\kappa_r - 1)$. This completes the proof.

It should be noticed that the values of these upper limits and hence of L depend only upon k and c.

The result which has thus been reached may be recapitulated in the following theorem.

THEOREM 6. *When $\sum|a_n + i\beta_n|$ is divergent, the continued fraction* (18) *will converge if the β-constituents alternate in sign and the ratio $|\beta_n| : |a_n|$ has a finite lower limit,* also if a_n never varies in sign and $|a_n| : |\beta_n|$ has such a limit.*

§ 15. When the second set of conditions of the last theorem are fulfilled and $|a_n| \geqq |\beta_n|$ for every value of n, the signs of the real parts of any convergent and hence of the continued fraction can be determined without difficulty. For consider the equations obtained by substituting K_i, $k_{a\gamma}^{(i)}$, $k_{\beta\gamma}^{(i)}$ for M_i^2, C_i, D_i in (13). It will be found that the only term which varies in sign is $2\beta_n k_{\beta\gamma}^{(n-1)}$, and the absolute value of this term is less than that of $2a_n k_{a\gamma}^{(n-1)}$. The following result will then be evident.

THEOREM 7. *If a_n has a constant sign and $|a_n| \geqq |\beta_n|$ for all values of n, the real part of any convergent and hence of the limit of continued fraction will have the same sign as a_1; if β_n alternates in sign and $|\beta_n| \geqq |a_n|$, the sign of the imaginary part is opposite to that of β_1.*

IV. *On a certain class of algebraic continued fractions.*

§ 16. Application of the results of section III. may now be made to algebraic continued fractions. Consider first any fraction of the form

* "*Untere Grenze*" (WEIERSTRASS); not necessarily a value which the ratio approaches when n increases indefinitely.

$$\frac{1}{d_1 z} + \frac{1}{d_2 z} + \frac{1}{d_3 z} + \cdots$$

in which the d_n are real numbers alternately positive and negative.

Theorem 6 is applicable only when $\sum |d_n|$ is divergent, and shows that the continued fraction converges over the entire plane of z with the exception of the whole or a part of the real axis. Theorem 2 proves that the roots of the numerators and denominators lie only upon the real axis. Above and below the axis the convergents are therefore analytic. That their limit is also an analytic function can be seen from the series (16). For since the individual terms of (16) are analytic, it suffices, by a well-known theorem of WEIERSTRASS,* to prove that the convergence is uniform. Take any circle (R) whose boundary lies wholly above or below the axis, and let $1/\bar{k}$ denote the smallest value of $|y|/|x|$ for this circle. Then since $|\alpha_n| \leqq \bar{k}|\beta_n|$ throughout (R), we need only to demonstrate the uniform convergence of $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$. If in §§ 13 and 14, k is replaced by $\bar{k}$, its maximum value within (R), the reasoning of these paragraphs will be seen to hold for every value of z within the circle or upon its boundary. The number L, which depends only on $\bar{k}$ and c, is moreover independent of z. If $|\bar{D}_1|$ denotes the smallest value of $|D_1|$ for the circle (R), the series

$$\frac{L}{\bar{k}+1} \sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$$

will be less, *term by term*, than a geometrical progression whose first term is $|\bar{D}_1^{-1}|$ and whose ratio c^{-1} is less than unity. It follows that $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$ is uniformly convergent. The continued fraction therefore represents an analytic function.

The position of the zeros of this analytic function can be quickly determined from theorem 2. For when a sequence of rational functions—in the present case the convergents—converge uniformly to an analytic function as their limit, the zeros of this function are the condensation-points of the zeros of the rational fractions.† The analytic function defined by the continued fraction therefore can vanish only for real values of z.

§ 17. When $\sum |d_n|$ is convergent, the even and the odd convergents define distinct functions. Theorem 1 shows that they converge over the entire plane with the possible exception of the real axis. Not only is this true, but in either sequence of convergents the numerators and denominators separately converge, and their limits are holomorphic functions. This may be demonstrated as follows:

* WEIERSTRASS, *Abhandlungen aus der Functionenlehre*, p. 73 ff.; or HARKNESS AND MORLEY, *Analytic Functions*, § 81.

† HURWITZ, Mathematische Annalen, vol. 33, p. 249.

The denominators of the convergents, also the numerators, contain alternately only odd and even powers of z, and in each one the coefficient of z^r, when it does not vanish, consists of certain products of the d_n taken r at a time. By the aid of equation (1) it can be easily shown that the products which figure in the coefficient of z^r in the denominator (numerator) of the nth convergent are included among the products which make up the corresponding coefficient of the second succeeding denominator (numerator). Hence as n is indefinitely increased, the coefficient of any power of z tends to take a definite limiting form. But by virtue of our initial hypothesis $\prod(1 + |d_n|z)$ converges for every finite value of z. It may therefore be expressed as a series $\sum c_r z^r$ which likewise converges over the finite plane, and the coefficient c_r is the limit of the sum of all the partial products of the $|d_n|$ taken r at a time. As c_r is obviously greater than the limit of the coefficient of z^r in either sequence of alternate numerators or denominators, the limits of the latter must be holomorphic functions.

It will be observed that *this conclusion always holds when* $\sum |d_n|$ *is convergent, irrespective of the conditions imposed upon* d_n *in our continued fraction.*

In the special case under discussion the position of the roots of the limiting functions can be inferred from theorem 2 in the same manner as when $\sum |d_n|$ was divergent. Theorem 7 may also be applied for a part of the plane.

Before summing up our results, it will be advantageous to transform the continued fraction so as to make the coefficients of the denominators of the partial quotients positive. We have then the following theorem.

THEOREM 8. *If any algebraic continued fraction has the form*

$$\frac{1}{b_1 z -}\,\frac{1}{b_2 z -}\,\frac{1}{b_3 z -}\cdots$$

and every element b_n *is positive, the continued fraction will have the following properties:*

(*a*) *The zeros of the numerators and denominators of its convergents all lie upon the real axis of* z.

(*b*) *If* $\sum |b_n|$ *is divergent, it represents a function which is analytic over the entire plane with the possible exception of the axis and which vanishes only for real values of* z.

(*c*) *If* $\sum |b_n|$ *is convergent, the limits of the numerators and denominators of the even, also of the odd convergents are holomorphic functions whose zeros lie upon the real axis.*

(*d*) *The imaginary part of each convergent or of the limiting function will have a sign opposite to the imaginary part of* z *in the two quarter planes for which* $|y| \geqq |x|$.

The continued fraction of the last theorem can readily be transformed, by neglecting a numerical factor, so as to take either the form

$$\frac{1}{a_1 z'} + \frac{1}{a_2 z'} + \frac{1}{a_3 z'} + \cdots \qquad (a_n > 0),$$

or the form preferred by STIELTJES and cited in § 1. To throw it into the first form one has only to put $z = iz'$. If it is to be reduced to the form of STIELTJES, first put $z = w^{-1}$ and transform so as to remove w from the denominators of the partial quotients. Each numerator after the first becomes equal to $-w^2$. The desired form is then immediately obtained by setting $-w^{-2} = z'$ and clearing the numerators of z'.

§ 18. The considerations by which theorem 8 was established apply with little change to continued fractions having the form

$$\frac{1}{b_1 z + c_1} + \frac{1}{-b_2 z - c_2} + \frac{1}{b_3 z + c_3} + \frac{1}{-b_4 z - c_4} + \cdots$$

in which the coefficients b_n have a common sign. If both $\sum b_n$ and $\sum c_n$ are convergent, $\prod (1 + |b_n z| + |c_n|)$ is also convergent and can be expressed as a power series which converges over the finite plane. As in § 17, the numerators and denominators of the alternate convergents tend to assume a definite limiting form, in which the coefficient of each power of z is the sum of certain products of the b_n and c_n whose moduli are included among the products which make up the coefficient of the corresponding power of z in the series for the above product $\prod$. The limits of the numerators and denominators of the alternate convergents are therefore again holomorphic functions.

If $\sum b_n$ or $\sum |c_n|$ is divergent, some additional restriction must be added. Suppose $|c_n| / |b_n|$ to have an upper limit L. Then at least $\sum b_n$ is divergent. Since

$$\frac{|\beta_n|}{|a_n|} = \frac{|yb_n|}{|xb_n + c_n|} \geqq \frac{|yb_n|}{|xb_n| + L|b_n|},$$

it is evident that $|\beta_n| / |a_n|$ has a finite lower limit for any value of z for which $y > 0$. The conclusions of (*b*), theorem 8, then follow. Hence

THEOREM 9. *If a continued fraction has the form*

$$\frac{1}{b_1 z + c_1} - \frac{1}{b_2 z + c_2} - \frac{1}{b_3 z + c_3} - \frac{1}{b_4 z + c_4} - \cdots$$

in which the coefficients b_n have a common sign, it will have the following properties:

(*a*) *The roots of the numerators and denominators of its convergents all lie upon the real axis of z.*

(*b*) *If $\sum b_n$ and $\sum |c_n|$ are convergent, the limits of the numerators and denominators of the odd, also of the even convergents are holomorphic functions whose zeros lie upon the real axis.*

(c) *If* $\sum b_n$ *is divergent and* $|c_n|/|b_n|$ *has an upper limit, the continued fraction converges for all values of z which are not real, and it represents a function which is analytic over the entire plane with the exception of the whole or a part of the real axis and which vanishes only for real values.*

The reality of the roots of the continued fraction has been previously proved by SYLVESTER,* who also shows that the roots of the denominators of two consecutive convergents alternate with each other. He does not consider the question of the convergence of the continued fraction when the number of elements is infinite.

§ 19. The theorems of III. may also be applied to certain classes of irregular continued fractions. It will suffice here to give as illustrations two applications of this nature.

THEOREM 10. *If the elements of an algebraic continued fraction*

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots$$

are all either positive numbers multiplied by z or complex numbers $a_n + i\beta_n$ *in which* $|a_n|/|\beta_n|$ *exceeds some fixed positive number for all values of n, the continued fraction will have the following properties:*

(a) *None of the roots of the numerators and denominators of its convergents will lie within the half plane in which the real part of z is positive, or upon its boundary.*

(b) *If* $\sum|\lambda_n|$ *is divergent, the continued fraction represents within this half plane an analytic function which does not vanish at any point in its interior.*

(c) *If* $\sum|\lambda_n|$ *is convergent, the limits of the numerators and denominators of the even, also of the odd convergents are holomorphic functions whose roots lie without this half plane.*

The second class of irregular algebraic continued fractions is that in which the even (odd) denominators of the partial quotients are all positive numbers and the odd (even) partial denominators are positive numbers or such numbers multiplied by z. The region of convergence covers the same half plane and the zeros of the numerators and denominators of the convergents and of the limiting functions lie upon the negative half of the axis. The continued fraction of STIELTJES is the special case in which z appears in *every* odd denominator..

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* Philosophical Transactions, vol. 143, part I, p. 497.